

BRAUER–SIEGEL THEOREM FOR ELLIPTIC SURFACES

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ABSTRACT. We consider higher-dimensional analogues of the classical Brauer–Siegel theorem focusing on the case of abelian varieties over global function fields. We prove such an analogue in the case of constant families of elliptic curves and abelian varieties.

*To our teachers V.E. Voskresenskiï and Yu.I. Manin
to their 80th and 70th birthdays, respectively*

1. INTRODUCTION

The classical Brauer–Siegel theorem, which is one of the milestones of the number theory of the past century, reflects deep connections between algebraic, arithmetical, analytic, and (in the function field case) geometric properties of global fields. Not only is the theorem a working tool in a variety of problems concerning number and function fields, but the underlying ideas have been recently put into much broader context expanding far beyond number theory (see, for example, [ST]).

Recall that the theorem describes the asymptotic behaviour of the product of two important arithmetic invariants of a number field K , the class number $h(K)$ and the regulator $R(K)$, as the discriminant $d(K)$ tends to infinity. More precisely, it says that the ratio $r = \log(hR)/\log(\sqrt{|d|})$ tends to 1 provided at least one of the following conditions is satisfied: 1) the degree $n = [K : \mathbb{Q}]$ remains the same for all K 's in the sequence of fields under consideration; 2) $n/\log(|d|)$ tends to 0 and all K 's are normal. Even in this not-so-effective form there

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are many useful applications. Some effective versions of the theorem are known in several particular cases (see [St] and references therein).

A natural question whether the statement of the theorem still holds when none of conditions 1) and 2) is satisfied, or under some weaker assumptions, remained widely open until recently. In the paper [TV2] there were obtained some asymptotic bounds on r generalizing the statement of the Brauer–Siegel theorem. These techniques, together with those of an earlier paper [TV1] led to a new concept of infinite global field which is an important object for further investigation. Combined with Weil’s “explicit formulae” (see [LT]), they yielded quite a few concrete arithmetic applications, like new estimates for regulators. Note that even more general approach was used in a recent paper [Zy] where the normality assumption on K was weakened.

The above mentioned results present the state of the art in the research area concentrated around the classical Brauer–Siegel theorem. In the present paper we make an attempt to treat some new problems arising from these achievements. Namely, one can think about higher dimensional analogues of the Brauer–Siegel theorem. In particular, if E is a commutative algebraic group defined over a global field K , one can define an analogue of the class number $h(E)$ and the regulator $R(E)$. Moreover, the classical analytical class number formula of Dirichlet admits higher dimensional analogues both for algebraic tori [Shyr] and, conjecturally, for abelian varieties (Birch and Swinnerton-Dyer). This motivates the study of asymptotic behaviour of $h(E)R(E)$ in appropriately chosen families of groups E when the “discriminant” $d(E)$ tends to infinity. In the case where E is an abelian variety, recent work of Hindry and Pacheco contains quite a new approach to this kind of asymptotic problems, both in the number field case [Hi] and in the function field case [HP]. This work was an additional motivation for publishing our results because the approach of Hindry and Pacheco is, in a sense, “orthogonal” to ours: loosely speaking, they consider “vertical” families of abelian varieties (say, in the function field case

the genus of the underlying curve X is fixed and the conductor of the abelian variety grows) while we consider “horizontal” families where the genus of X tends to infinity.

2. MAIN THEOREM

We fix the ground field $k = \mathbb{F}_q$ and consider a (smooth, projective, geometrically irreducible) curve X/\mathbb{F}_q of genus g . Let $K = \mathbb{F}_q(X)$, and let E/K be a (smooth, connected) commutative algebraic K -group. Our goal is to study asymptotic behaviour of the “class number” $h(E)$ as $g \rightarrow \infty$. In the present paper we focus on the particular case where $E = A$ is an abelian variety (see, however, Section 3 for the case where E is an algebraic K -torus). Let $\text{III} := |\text{III}(A)|$ be the order of the Shafarevich–Tate group of A , and Δ the determinant of the Mordell–Weil lattice of A (cf. [Mi], [Hi]). In this section we consider the most trivial “constant” case, i.e. $E \cong E_0 \times_{\mathbb{F}_q} K$ where E_0 is an \mathbb{F}_q -group; see Section 3 for a more general setting.

To state our main result, we recall some notation from [TV1]. If the ground curve $X = X_0$ varies in a family $\{X_i\}$, we denote by g_i the genus of X_i ($g_i \rightarrow \infty$), by $N_m(X_i)$ the number of \mathbb{F}_{q^m} -points of X_i , and we always assume that for every $m \geq 1$ there exists a limit $\beta_m := \lim_{i \rightarrow \infty} \frac{N_m(X_i)}{g_i}$. Such families are called *asymptotically exact*; any family contains an asymptotically exact subfamily; any tower (i.e. a family such that $k(X_i) \subset k(X_{i+1})$ for every i) is asymptotically exact; see [Ts], [TV1] for more details. We shall often drop the index i if this does not lead to confusion.

Theorem 2.1. *Let $E = E_0 \times_{\mathbb{F}_q} K$ where E_0 a fixed elliptic \mathbb{F}_q -curve. Let K vary in an asymptotically exact family, and let β_m be the corresponding limits. Then*

$$\lim_{i \rightarrow \infty} \frac{1}{g_i} \log_q(\text{III} \cdot \Delta) = 1 - \sum_{m=1}^{\infty} \beta_m \log_q \frac{N_m(E_0)}{q^m},$$

where $N_m(E_0) = |E_0(\mathbb{F}_{q^m})|$.

Proof. Denote by ω_j ($j = 1, \dots, 2g$) the eigenvalues of Frobenius acting on $H^1(X)$, and by ψ_1, ψ_2 the eigenvalues of Frobenius acting on $H^1(E_0)$. We have $\omega_j \bar{\omega}_j = \psi_1 \psi_2 = q$.

Put $t = q^{-s}$ and consider the Hasse–Weil L -function of E/K . According to the Birch and Swinnerton-Dyer conjecture (which, under our hypotheses, is a theorem [Mi], [Oe]), the value of $L_E(t)/(1-qt)^r$ at $t = q^{-1}$ equals $q^{1-g} \cdot \text{III} \cdot \Delta / [\#E_0(k)]^2$. Here r is the rank of $E(K)/E(K)_{\text{tors}}$; this number is equal to the number of pairs (i, j) such that $\psi_i = \omega_j$ (loc. cit.). This gives us Milne’s formula

$$\text{III} \cdot \Delta = q^g \prod_{\omega_j \neq \psi_i} \left(1 - \frac{\psi_i}{\omega_j}\right).$$

It is convenient to put $\psi_i = \alpha_i \sqrt{q}$, $\omega_j = \gamma_j \sqrt{q}$, and, taking into account that the Frobenius roots can be written as conjugate pairs, to write the above formula as

$$(1) \quad \text{III} \cdot \Delta = q^g \prod_{\alpha_i \neq 1/\gamma_j} (1 - \alpha_i \gamma_j).$$

Set $\alpha_1 = \alpha$, $\alpha_2 = \bar{\alpha}$. First consider the case where $r = 0$. Then the right-hand side of (1) can be written as $q^g P_X(\alpha/\sqrt{q}) P_X(\bar{\alpha}/\sqrt{q})$, where $P_X(t)$ is the numerator of the zeta-function of X :

$$Z_X(t) = \frac{P_X(t)}{(1-t)(1-qt)}.$$

Hence the right-hand side of (1) equals

$$q^g \left[\left(1 - \frac{\alpha}{\sqrt{q}}\right) (1 - \alpha \sqrt{q}) Z_X \left(\frac{\alpha}{\sqrt{q}}\right) \left(1 - \frac{\bar{\alpha}}{\sqrt{q}}\right) (1 - \bar{\alpha} \sqrt{q}) Z_X \left(\frac{\bar{\alpha}}{\sqrt{q}}\right) \right].$$

We now write $Z_X(t) = \prod_{m=1}^{\infty} (1-t^m)^{-\beta_m}$, then we have $\beta_m = \lim_{g \rightarrow \infty} \frac{B_m}{g}$ (by our assumption, the limit exists), and we get

$$\begin{aligned} \lim_{g \rightarrow \infty} \frac{1}{g} \log_q (\text{III} \cdot \Delta) &= 1 + \log_q \left(\prod_{m=1}^{\infty} \left(1 - \frac{\alpha^m}{q^{\frac{m}{2}}}\right)^{-\beta_m} \left(1 - \frac{\bar{\alpha}^m}{q^{\frac{m}{2}}}\right)^{-\beta_m} \right) \\ &= 1 - \sum_{m=1}^{\infty} \beta_m \log_q \left(1 + \frac{1}{q^m} - \frac{\alpha^m + \bar{\alpha}^m}{q^{\frac{m}{2}}}\right) \\ &= 1 - \sum_{m=1}^{\infty} \beta_m \log_q \frac{N_m}{q^m} \end{aligned}$$

(here $N_m = |E_0(\mathbb{F}_{q^m})|$, and the last equality follows from the Weil formula). Note that the series on the right-hand side converges according to [Ts]. Indeed, we know that the series $\sum_{m=1}^{\infty} \frac{m\beta_m}{q^{\frac{m}{2}}-1}$ converges [Ts, Cor.1]. We have $\frac{N_m}{q^m} = 1 + q^{-m} - \frac{\alpha^m + \bar{\alpha}^m}{q^{\frac{m}{2}}}$. Fix $m > m_0$ big enough. Put $x = \frac{\alpha^m + \bar{\alpha}^m}{q^{\frac{m}{2}}} - q^{-m}$. Since $|\alpha^m + \bar{\alpha}^m| \leq 2$, we have

$$\left| \log_q \frac{N_m}{q^m} \right| = |\log_q(1 - x)| \leq c \sum_{n=1}^{\infty} (q^{-\frac{m}{2}})^n \leq c' q^{-\frac{m}{2}} \leq c' \frac{m}{q^{\frac{m}{2}} - 1}.$$

Hence the series $\sum \beta_m \log_q \frac{N_m}{q^m}$ converges.

Let us now consider the case where $r > 0$. Our key observation is that as $g \rightarrow \infty$, the rank cannot grow as fast as g , i.e., we always have $\lim_{g \rightarrow \infty} \frac{r}{g} = 0$.

Indeed, if $\lim_{g \rightarrow \infty} \frac{r}{g} = c > 0$, then there is at least one multiple Frobenius root $\omega_j = \psi_1$ or ψ_2 with multiplicity $\geq cg$. Hence the Weil measure (cf. [TV1])

$$\mu_{\Omega} = \frac{1}{g} \sum_{j=1}^{2g} \delta_{\gamma_j} \quad (\text{where } \delta_{\gamma_j} \text{ is the Dirac measure})$$

tends (as $g \rightarrow \infty$) to a measure that is greater than or equal to $c\delta_{\gamma_j}$. But according to [TV1, Th.2.1], the limit measure $\mu = \lim_{g \rightarrow \infty} \mu_{\Omega}$ must have a continuous density, contradiction.

(As pointed out by the referee, this observation might happen to be deducible from the “explicit formulae” for elliptic curves, see, e.g., [Br].)

Thus, in the general case where $r > 0$, we get the required result as follows.

Let us introduce an auxiliary function $\delta(g) = 1 + \varepsilon(g)$ such that $\lim_{g \rightarrow \infty} \varepsilon(g) = 0$ and $\lim_{g \rightarrow \infty} \left(\frac{r \log \varepsilon(g)}{g} \right) = 0$. Let

$$F(g) = q^g P_X(\delta(g)\alpha/\sqrt{q}) P_X(\delta(g)\bar{\alpha}/\sqrt{q}).$$

We have, on the one hand,

$$(2) \quad \lim_{g \rightarrow \infty} \frac{\log_q F(g)}{g} = 1 - \sum_{m=1}^{\infty} \beta_m \log_q \left(\frac{N_m}{q^m} \right),$$

and, on the other hand,

$$\lim_{g \rightarrow \infty} \frac{1}{g} \log_q F(g) = \lim_{g \rightarrow \infty} \left(\frac{1}{g} \log_q (\text{III} \cdot \Delta) \right).$$

To prove the last equality, we write

$$\begin{aligned} F(g) &= q^g \prod_{j=1}^{2g} (1 - \alpha \gamma_j \delta(g))(1 - \bar{\alpha} \gamma_j \delta(g)) = \delta(g)^{4g} q^g \prod_{j=1}^{2g} \left(\frac{1}{\delta(g)} - \alpha \gamma_j \right) \left(\frac{1}{\delta(g)} - \bar{\alpha} \gamma_j \right) \\ &= \delta(g)^{4g} q^g \prod_{\gamma_j = 1/\alpha} \left(\frac{1}{\delta(g)} - \alpha \gamma_j \right) \left(\frac{1}{\delta(g)} - \bar{\alpha} \gamma_j \right) \cdot \prod_{\gamma_j \neq 1/\alpha} \left(\frac{1}{\delta(g)} - \alpha \gamma_j \right) \left(\frac{1}{\delta(g)} - \bar{\alpha} \gamma_j \right) \\ &= \delta(g)^{4g} \left(\frac{1}{\delta(g)} - 1 \right)^r \cdot q^g \cdot \prod_{\gamma_j \neq 1/\alpha} (1 - \alpha \gamma_j \delta(g))(1 - \bar{\alpha} \gamma_j \delta(g)) \cdot \frac{1}{\delta(g)^{4g-r}} \\ &= (1 - \delta(g))^r \cdot q^g \cdot \prod_{\gamma_j \neq 1/\alpha} (1 - \alpha \gamma_j \delta(g))(1 - \bar{\alpha} \gamma_j \delta(g)). \end{aligned}$$

Hence

$$\begin{aligned} \lim_{g \rightarrow \infty} \frac{1}{g} \log_q F(g) &= \lim_{g \rightarrow \infty} \left(\frac{1}{g} \log_q (1 - \delta(g))^r \right) + \lim_{g \rightarrow \infty} \left(\frac{1}{g} \log_q (\text{III} \cdot \Delta) \right) \\ &= \lim_{g \rightarrow \infty} \frac{r \log_q \varepsilon(g)}{g} + \lim_{g \rightarrow \infty} \left(\frac{1}{g} \log_q (\text{III} \cdot \Delta) \right) \\ &= \lim_{g \rightarrow \infty} \left(\frac{1}{g} \log_q (\text{III} \cdot \Delta) \right). \end{aligned}$$

Note that the series

$$\sum_{m=1}^{\infty} \beta_m \log_q \left(1 + \frac{1}{q^m} + \frac{(\delta(g)\alpha)^m + (\delta(g)\bar{\alpha})^m}{q^{\frac{m}{2}}} \right)$$

converges for every fixed $\delta(g)$ sufficiently close to 1. Hence the passage to the limit in (2) is legitimate. \square

A direct analogue of Theorem 2.1 is true for constant abelian varieties of arbitrary dimension.

Theorem 2.2. *Let $A = A_0 \times_{\mathbb{F}_q} K$ where A_0 a fixed abelian \mathbb{F}_q -variety of dimension d . Let K vary in an asymptotically exact family, and let β_m be the corresponding limits. Then*

$$\lim_{i \rightarrow \infty} \frac{1}{dg_i} \log_q (\text{III} \cdot \Delta) = 1 - \sum_{m=1}^{\infty} \beta_m \log_q \frac{N_m(A_0)^{1/d}}{q^m},$$

where $N_m(A_0) = |A_0(\mathbb{F}_{q^m})|$.

Proof. The proof goes as for elliptic curves, *mutatis mutandis*. The value of $L_A(t)/(1 - qt)^r$ at $t = q^{-1}$ equals $q^{1-dg} \cdot \text{III} \cdot \Delta / (\#A_0(k)) \cdot$

$\#A_0^\vee(k))$, where A_0^\vee stands for the dual abelian variety. According to [Mi, Th. 3], this leads to a formula similar to (1)

$$\text{III} \cdot \Delta = q^{dg} \prod_{\alpha_i \neq 1/\gamma_j} (1 - \alpha_i \gamma_j),$$

where α_i ($i = 1, \dots, 2d$) are the (normalized) Frobenius roots of A_0 . Therefore the case $r_A = 0$ is treated, word for word, as in the case $d = 1$. If $r_A > 0$, we have to prove that $\frac{r_A}{g_X} \rightarrow 0$ as $g_X \rightarrow \infty$, and then apply the same argument as for elliptic curves. Assume the contrary, i.e., $\lim_{g_X \rightarrow \infty} \frac{r_A}{g_X} = c > 0$. Note that the Mordell–Weil group $A(K)/A(K)_{\text{tors}}$ is isomorphic to $\text{Hom}_k(J_X, A_0)$. This implies that at least one Frobenius root of J_X (or of X , which is the same) appears with the multiplicity proportional to g . As in the one-dimensional case, we then consider the Weil measure μ_Ω and see that its limit as $g \rightarrow \infty$ has discontinuous density which contradicts [TV1].

The theorem is proved. \square

3. GENERALIZATIONS

In this section we shall describe some possible generalizations of Theorem 2.1. To make our approach more clear, we shall first restrict ourselves to considering the case where E is an elliptic K -curve. Denote by \mathcal{E} the corresponding elliptic surface (this means that there is a proper connected smooth morphism $f: \mathcal{E} \rightarrow X$ with the generic fibre E). Assume that f fits into an infinite Galois tower, i.e. into a commutative diagram of the following form:

$$(3) \quad \begin{array}{ccccccc} \mathcal{E} = \mathcal{E}_0 & \longleftarrow & \mathcal{E}_1 & \longleftarrow & \dots & \longleftarrow & \mathcal{E}_j & \longleftarrow & \dots \\ \downarrow f & & \downarrow & & & & \downarrow & & \\ X = X_0 & \longleftarrow & X_1 & \longleftarrow & \dots & \longleftarrow & X_j & \longleftarrow & \dots, \end{array}$$

where each lower horizontal arrow is a Galois covering. Let us introduce some notation. For every $v \in X$, let $E_v = f^{-1}(v)$, let $r_{v,i}$ denote the number of points of X_i lying above v , $\beta_v = \lim_{i \rightarrow \infty} r_{v,i}/g_i$ (we suppose the limits exist). Furthermore, denote by $f_{v,i}$ the residue degree of a point of X_i lying above v (the tower being Galois, this does not depend

on the point), and let $f_v = \lim_{i \rightarrow \infty} f_{v,i}$. If $f_v = \infty$, we have $\beta_v = 0$. If f_v is finite, denote by $N(E_v, f_v)$ the number of $\mathbb{F}_{q^{f_v}}$ -points of E_v . Finally, let τ denote the “fudge” factor in the Birch and Swinnerton-Dyer conjecture (see [Ta] for its precise definition). Under this setting, we dare formulate the following

Conjecture 3.1. *Assuming the Birch and Swinnerton-Dyer conjecture for elliptic curves over function fields, we have*

$$\lim_{g \rightarrow \infty} \frac{1}{g} \log_q(\text{III} \cdot \Delta \cdot \tau) = 1 - \sum_{v \in X} \beta_v \log_q \frac{N(E_v, f_v)}{q^{f_v}}.$$

Remark 3.2. One can check that in the constant case Conjecture 3.1 is consistent with Theorem 2.1. The first nontrivial case to be considered is that of an isotrivial elliptic surface.

Here are some questions for further investigation.

Question 3.3. How can one formulate an analogue of Conjecture 3.1 for more general towers when diagram (3) does not commute? for more general families when there are no upper horizontal arrows in diagram (3)?

With an eye towards even further generalizations of the Brauer–Siegel theorem to arbitrary commutative algebraic groups, the next extreme case to be considered is that of algebraic tori. In that case the analogues of the class number and the regulator are known [Ono], [Vo]. Moreover, there is an analogue of the analytic class number formula of Dirichlet established in [Shyr] for tori over number fields. Together with Theorem 2.2, this motivates the following

Conjecture 3.4. *Let $T = T_0 \times_{\mathbb{F}_q} K$, where T_0 is a fixed \mathbb{F}_q -torus. Then*

$$\lim_{g \rightarrow \infty} \frac{1}{g} \log h(T) = \lim_{g \rightarrow \infty} \frac{1}{g} \log \sqrt{\mathcal{D}_T} - \sum_{m=1}^{\infty} \beta_m \log_q \frac{N_m(T_0)}{q^{md}},$$

where $d = \dim T$, $N_m(T_0) = |T_0(\mathbb{F}_{q^m})|$, \mathcal{D}_T is the “quasi-discriminant” of T (cf. [Shyr]), and all other notation is as in the previous sections.

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